ALGEBRA IV FINAL EXAMINATION

Attempt all questions. Total Marks: 50. If you use a result proved in class then it is enough to just quote it. This is a three hour exam. You will also get some time before and after the examination for downloading and uploading respectively.

(1) Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$. Identify a splitting field $K/\mathbb{Q}$, and the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of f over $\mathbb{Q}$. Write all the intermediate subfields L with $\mathbb{Q} \subset L \subset K$, and the corresponding subgroups of the Galois group. (10 marks)

(2) What is the Galois group of an irreducible degree 4 polynomial $f(x)$ over $\mathbb{Q}$ whose resolvent $r(x)$ is $x^3 - 3x + 1$? What is the Galois group of $x^5 - 4x + 2$ over $\mathbb{Q}$? Justify both answers. (5+5=10 marks)

(3) Is the field extension $\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}$ a radical extension? Justify your answer. (10 marks)

(4) Let $\Phi_n(x)$ be the nth cyclotomic polynomial. Is $\Phi_{18}(x)$ irreducible over (a) $\mathbb{F}_{23}$, (b) $\mathbb{F}_{43}$? Justify your answers. (5+5 = 10 marks)

(5) Let F be a field, let F^{al} be an algebraic closure of F. Let σ be an element of $\mathrm{Gal}(F^{al}/F) = \mathrm{Aut}_F(F^{al})$. Let K be the fixed field of σ. Prove that any finite extension L of K is Galois, and the Galois group $\mathrm{Gal}(L/K)$ is cyclic. (10 marks)